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On a Binomial Model of Option Pricing

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Abstract: The conventional models of option pricing are based on the artificial proposition of the market neutrality to the risk, which is equivalent to the assumption that the expected return μ equals the risk-free return r. However, in real markets μ may significantly differ from r. This may lead to the noticeable difference between model and real values of option prices. We consider here a modification of the Cox-Ross-Rubenstein binomial model that takes into account the dependence of option prices on μ.

Keywords: Option pricing; Forecasting; Binomial model.

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1 Introduction

The conventional models for the determination of the option prices [1, 2, 3, 4, 5, 6] are based on the assumption of market neutrality to the risk. In particular, the famous Black-Scholes option pricing formula:

$$C_{BS} = S \cdot \Phi\left(\frac{\ln\frac{S}{X} + rT + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right) - X\exp(-rT)\cdot\Phi\left(\frac{\ln\frac{S}{X} + rT - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right), \tag{1}$$

was obtained for the continuous model [8] and for the discrete binomial model [9, 10, 11, 12] under the assumption that the expected return μ of the underlying assets equals the risk-free return r.

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Here C_{BS} is Black-Scholes option price, S is the stock price, σ is the volatility of the stock price per annum, X is the option exercise price, T is the option expiration date (maturity), and

$$\Phi(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} \exp(-\frac{S^2}{2}) dS$$

is the standard normal cumulative distribution function.

However, in a real risky market μ may significantly differ from r. As a result, we may have a noticeable difference between the predicted and real values of the option prices [7].

We introduce a modification of the Cox-Ross-Rubenstein discrete binomial model [9] taking into account the dependence of option prices on μ. Then we realize the transition to the continuous limit when $\mu \neq r$ by extending the de Moivre-Laplace theorem.

2 Modification of the Cox-Ross-Rubenstein model: $\mu \neq r$ case

We need a modification of the Cox-Ross-Rubenstein model to include the case $\mu \neq r$. In order to determine the option price we assume that:

$$C = d(T) \cdot E(\max(S(T) - X, 0)), \tag{2}$$

where $d(T)$ is the discount factor and E is the operator of mathematical expectation (without any assumptions).

Following [9], we divide the time interval [0,T] into n subintervals with duration $\Delta t = \frac{T}{n}$. During each subinterval, the stock price may grow by a factor $u_n > 1$ with probability p_n, or to diminish by a factor $d_n < 1$ with probability $q_n = 1 - p_n$. We also assume that u_n, d_n, p_n depend neither on the location of the subinterval $[m\,\Delta t, (m+1)\,\Delta t]$, nor on the previous change of stock price. This actually means that the stock price change is described by a stationary Markov chain.

For this model the call option price C_n satisfies the binomial law [9]:

$$C_n = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \cdot \frac{p_n^m\, q_n^{n-m}}{(1 + r\frac{T}{n})^n} \cdot \max\{0, S u_n^m d_n^{n-m} - X\}, \tag{3}$$

where $S u_n^m d_n^{n-m}$ is the stock price to the option execution date in the case of m times price increase and $(n-m)$ times price decrease. The values of u_n, d_n, p_n are found from the continuous Samuelson model:

$$dS = \mu\, S(t)\, dt + \sigma\, S(t)\, dz(t), \tag{4}$$

where $S(t)$ is the stock price at time t, $dz(t)$ is the stochastic differential of the normalized Wiener process with zero mean and variance dt.

In order that formulae Eq (3) and Eq (4) be compatible it is that the following conditions are fulfilled:

$$\lim_{n\to\infty} n\,(p_n\, \ln\frac{u_n}{d_n} + \ln d_n) = (\mu - \frac{\sigma^2}{2})\,T, \quad \lim_{n\to\infty} n\, p_n\, q_n\, \ln^2\left(\frac{u_n}{d_n}\right) = \sigma^2\, T. \tag{5}$$

Equations Eq (5) are satisfied if we assume:

$$\begin{aligned} u_n &= 1 + \sigma\sqrt{\frac{T}{n}} + \frac{\sigma^2}{2}\cdot\frac{T}{n}, \quad d_n = 1 - \sigma\sqrt{\frac{T}{n}} + \frac{\sigma^2}{2}\cdot\frac{T}{n}, \\ p_n &= \frac{1}{2} + \frac{1}{2}\left(\frac{\mu}{\sigma} - \frac{\sigma}{2}\right)\cdot\sqrt{\frac{T}{n}}, \quad q_n = 1 - p_n. \end{aligned} \tag{6}$$

Let us denote by m^* the least integer m for which the inequality $Su_n^m d_n^{n-m} \geq X$ is fulfilled. First, let assume that condition $\mu = r$ is justified. Then, from Eq (3) and Eq (6) follows the equality:

$$C_n = \sum_{m=m^*}^{n} \frac{n!}{m!(n-m)!} \cdot \left(S\hat{p_n}^m \hat{q_n}^{n-m} - X \frac{\tilde{p_n}^m \tilde{q_n}^{n-m}}{(1+r\frac{T}{n})^n} \right), \quad \text{where} \tag{7}$$

$$\tilde{p_n} = \frac{1}{2} + \frac{1}{2}\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\sqrt{\frac{T}{n}}, \quad \tilde{q_n} = 1 - \tilde{p_n},$$

$$\hat{p_n} = \frac{p_n u_n}{1 + r\frac{T}{n}} = \frac{\frac{1}{2} + \frac{1}{4}\sigma\sqrt{\frac{T}{n}} + \frac{r}{2\sigma}\sqrt{\frac{T}{n}} + \frac{1}{2}r\frac{T}{n} + \frac{r\sigma}{4}\left(\frac{T}{n}\right)^{\frac{3}{2}} - \frac{\sigma^3}{8}\left(\frac{T}{n}\right)^{\frac{3}{2}}}{1 + r\frac{T}{n}},$$

$$\hat{q_n} = \frac{q_n d_n}{1 + r\frac{T}{n}} = \frac{\frac{1}{2} - \frac{1}{4}\sigma\sqrt{\frac{T}{n}} - \frac{r}{2\sigma}\sqrt{\frac{T}{n}} + \frac{1}{2}r\frac{T}{n} - \frac{r\sigma}{4}\left(\frac{T}{n}\right)^{\frac{3}{2}} + \frac{\sigma^3}{8}\left(\frac{T}{n}\right)^{\frac{3}{2}}}{1 + r\frac{T}{n}}, \quad \hat{q_n} = 1 - \hat{p_n}.$$

Thus, due to the integral theorem of Moivre-Laplace, we get $C = \lim_{n\to+\infty} C_n$, where C is determined by the Black-Scholes option pricing formula Eq (1).

In the general case, when condition $\mu = r$ is violated, we have:

$$C_n = \sum_{m=m^*}^{n} \frac{n!}{m!(n-m)!} \cdot p_n^m q_n^{n-m} \cdot \left(Su_n^m d_n^{n-m} - X\right) \cdot \frac{1}{(1+r\frac{T}{n})^n}, \tag{8}$$

where p_n, q_n, u_n, d_n are determined by equations Eq (6). Therefore, from the Moivre-Laplace integral theorem, we have:

$$\sum_{m=m^*}^{n} \frac{n!}{m!(n-m)!} \cdot p_n^m q_n^{n-m} \to \Phi\left(\frac{\ln\frac{S}{X} + \mu T - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right). \tag{9}$$

Thus, in order to find the $\lim_{n\to+\infty} C_n$, where C_n is determined by equation Eq (8), it is enough to find the limit:

$$\lim_{n\to+\infty} \frac{1}{(1+r\frac{T}{n})^n} \sum_{m=m^*}^{n} \frac{n!}{m!(n-m)!} \cdot (p_n u_n)^m \cdot (q_n d_n)^{n-m}. \tag{10}$$

As the inequalities $p_n u_n + q_n d_n \neq 1$ and $p_n u_n + q_n d_n \neq 1 + r\frac{T}{n}$ are fulfilled for $\mu \neq r$, it is not possible to apply the local theorem of de Moivre-Laplace for the limit Eq (10). Therefore, we shall extend the local theorem of de Moivre-Laplace, as:

Theorem 1 *The following asymptotic expression holds for large n:*

$$\frac{n!}{m!(n-m)!} \cdot \bar{p_n}^m \bar{q_n}^{n-m} = \frac{1}{\sqrt{2\pi n \ddot{p_n}\ddot{q_n}}} \exp(\mu T)\exp(-\frac{\ddot{x_m}^2}{2})\left(1 + O(\frac{1}{\sqrt{n}})\right), \tag{11}$$

where

$$\ddot{p_n} = \frac{1}{2} + \frac{1}{2}\left(\frac{\mu}{\sigma} + \frac{\sigma}{2}\right)\sqrt{\frac{T}{n}}, \quad \ddot{q_n} = 1 - \ddot{p_n},$$

$$\ddot{x_m} = \frac{m - n\cdot\ddot{p_n}}{\sqrt{n\ddot{p_n}\ddot{q_n}}}, \quad \bar{p_n} = p_n u_n, \quad \bar{q_n} = q_n d_n.$$

Theorem 2

$$\lim_{n\to+\infty}\sum_{m=m^*}^{n}\frac{n!}{m!(n-m)!}\cdot\bar{p_n}^{m}\bar{q_n}^{n-m}=\exp(\mu T)\cdot\Phi\left(\frac{\ln\frac{S}{X}+\mu T+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right).$$

Applying theorem 2 and Eq (9) we can determine the limit value $C=\lim_{n\to+\infty}C_n$ of the call option price in the framework of the discrete binomial model:

$$C=\exp(-r\,T)\cdot\left[\exp(\mu\,T)\cdot\Phi\left(\frac{\ln\frac{S}{X}+\mu T+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)-X\cdot\Phi\left(\frac{\ln\frac{S}{X}+\mu T-\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)\right]. \quad (12)$$

For instance, assuming condition $\mu=r$ we get the Black-Scholes option pricing formula Eq (1).

Formula Eq (12) determines the call option price in the limit of the continuous Samuelson model Eq (4) and has been previously obtained in our work [7] on modification of the Black-Scholes formula for $\mu\neq r$. We have shown [7] moreover, that when, condition $\mu=r$ is violated, the call option price calculated by the Black-Scholes option pricing formula Eq (1) may give a noticeable bias.

3 Conclusions

Using the assumption $\mu\approx 2r$ and typical parameters (see below), we calculated the option prices applying the Black-Scholes formula Eq (1) and formula Eq (12) for C and obtained a systematic and essential difference between C_{BS} and C. This difference remains on the same level when the discount factor $d(T)=\exp(-rT)$ in Eq (12) changes to $\exp(-\mu T)$.

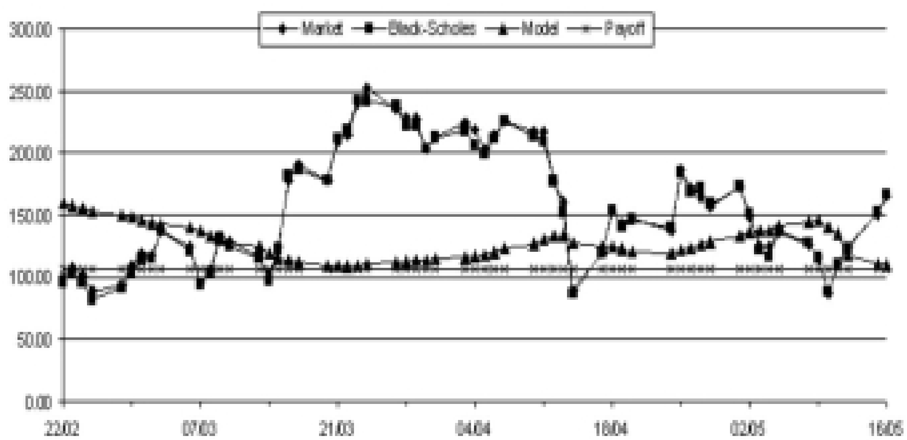

Figure 1: Dynamics of call option payoffs for the S&P500 index, for market option prices, for option prices determined by the Black-Scholes model and by model Eq (12) for the period 22.02.2000–16.05.2000.

The following parameters were used in our calculations: $r=0.08$, $\mu=0.16$, $\sigma=0.2$, $S=100$, $X=102$, $T=0.25$. Using the Black-Scholes formula we obtained $C_{BS}=4.00$. Using formula Eq (12) with $d(T)=\exp(-rT)$ we obtained $C=5.13$, which differs from C_{BS} by 28%. If we use $d(T)=\exp(-\mu T)$, formula Eq (12) gives 5.02, which differs from C_{BS} by 25.5%.

We compare in Figure 1 the actual option price (the payoff for the S&P500 index) at the expiration date T, the market option price, the option price corresponding to the Black-Scholes model and the option price obtained from our model Eq (12). The call option payoffs were calculated using the formula $P = \max[S(T) - X, 0]$. The values of the expected return μ in Eq (12) were calculated on the basis of the daily closing prices of basic stocks using the least squares method.

We see in Figure 1 that the market prices and the prices corresponding to the Black-Scholes model coincide in most presented time. However, they significantly differ from the call option payoffs. On the other hand, the estimation of option prices by formula Eq (12) is more close to call option payoffs compared with the market option prices and the option prices corresponding to the Black-Scholes model. Similar results were obtained for other options.

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